Algebraic Geometry – Research Project

The successful candidate will work in the scope of the PRIN project CuRVI " Curves, Ricci flat Varieties and their Interactions", in the broad area of Algebraic Geometry. Depending on the skills of the best candidate, they will be given one of the following problems, to be solved in collaborations with members of the group, specifically Bologna's unit (which includes Andreas Hochenegger and Elena Martinengo as external members, and Enrico Fatighenti as future member):

R3: Lagrangian fibrations on Irreducible symplectic varieties (ISV):

Lagrangian fibrations on ISVs can be constructed in two ways: the first is given by taking families of curves and their compactified Jacobians (or Prym varieties), as studied in a different task of the PRIN project (by the units of Bologna and Rome). A classical example in this setting is given by taking a complete linear system of curves in a symplectic surface and the associated compactified Jacobians of a given degree. The total space can be described as a moduli space of sheaves on the symplectic surface, and, depending on the degree of the Jacobians, it is either smooth or singular. The relations among these Lagrangian fibrations have already been exploited in [PR] and [MRS] to determine the topology of the associated moduli spaces. The second technique to construct examples is given by taking Fano varieties of K3 type and considering their hyperplane sections. If these hyperplane sections have an odd cohomology of level one, it is possible to construct a fibration over the space of hyperplanes with fibres the intermediate Jacobians of the hyperplane sections. An example of this is given by cubic fourfolds and the cubic threefolds they contain. Here the results of task S2 will be fundamental.
Moreover, we will investigate whether the existence of a Lagrangian fibration could be a property that is invariant under derived equivalence. This happens for Lagrangian (i.e. elliptic) fibrations on K3 surfaces, and the techniques which can be used in higher dimensions are related to the ones presented in S4. The aim of this part of the project is to construct more of these fibrations, by using both techniques, and directly by trying to compactify notable families of curves.

S2: Fano Varieties of K3 type:
Fano varieties with a special K3 structure attached to them are usually called Fano varieties of K3 type. This K3 like structure can be seen either as a Hodge substructure which looks like the Hodge structure of a K3 surface (or, more generally, the Hodge structure of a symplectic surface) or as a special subcategory of the derived category of the Fano variety which behaves like the derived category of a K3 surface in terms of the dualizing sheaf.
Up to now very few examples of Fano varieties of K3 type exist: beside the known examples given by cubic fourfolds, there are some linear and quadratic sections of Grassmannians and a couple more examples studied in [FM] and [BFMT2]. The goal of this part of the project is twofold: on the one hand, our first goal is to produce more examples of such varieties. The team is already working on a more efficient computation of cohomology for subvarieties of homogeneous varieties, and an exhaustive search of such varieties as zero loci of sections of homogeneous vector bundles is currently being implemented along the lines of [DBFT], providing partial classifications of Fano varieties in this setting. Possible generalizations of this framework are degeneracy loci of morphisms between vector bundles over homogeneous varieties, or more generally *orbital degeneracy loci*, a wide class of varieties which has been recently introduced and studied in [BFMT1].

On the other hand, to every example of Fano variety of K3 type we aim to explicitly associate a family of IHS varieties: as a moduli space of objects in the K3 category, if we can produce stability conditions in it, or with explicit geometry otherwise, like in the case of the families of Intermediate Jacobians on cubic fourfolds. Moreover, in the case of Fano fourfolds of K3 type an interesting open problem is also given by their rationality and stable rationality, which can often be formulated as problems of algebraicity on the associated IHS varieties. For instance, the (conjectural) rationality of a cubic fourfold is related to the birational geometry of its variety of lines F, and its stable rationality is related to algebraicity of a codimension two class in FxF. Partial progress in this direction is [MO].

S4: Derived categories of Irreducible symplectic varieties

Firstly we will study autoequivalences of derived categories of ISVs, in order to generalize deep results about K3 surfaces to them. Seidel and Thomas showed that a chain of spherical objects on a K3 surface gives rise to autoequivalences of its derived category, the so-called spherical twists, which generate a braid subgroup inside the group of autoequivalences. This gives a strong evidence to the homological mirror symmetry conjecture by Kontsevich. In [HK], a key result of the work by Seidel and Thomas was generalized to P-objects, which are the correct generalization of spherical objects for hyperkähler manifolds. We aim to construct natural configurations of P-objects on ISVs and investigate the subgroups generated by the corresponding P-twists inside the group of autoequivalences. The starting point will be the case of varieties of the form Hilb^n(K3), where we want to describe the P-objects also in terms of their geometry. Using deformation theory, we expect to generalize these descriptions also to other ISVs. Secondly, we aim to focus on the relations between derived equivalence, L-equivalence and birationality in the setting of ISV, and to study more in detail what happens for moduli spaces of objects on symplectic surfaces, thus generalizing the results of [MMY].

Bibliography:

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